Dynamical symmetry and analytical solutions of the non-autonomous quantum master equation of the dissipative two-level system: decoherence of quantum register

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Based on the non-autonomous quantum master equation, we investigate the dissipative and decoherence properties of the two-level atom system interacting with the environment of thermal quantum radiation fields. For this system, by a novel algebraic dynamic method, the dynamical symmetry of the system is found, the quantum master equation is converted into a Schrödinger-like equation and the non-Hermitian rate (quantum Liouville) operator of the master equation is expressed as a linear function of the dynamical u(2) generators. Furthermore, the integrability of the non-autonomous master equation has been proved for the first time. Based on the time-dependent analytical solutions, the asymptotic behavior of the solution has been examined and the approach to the equilibrium state has been proved. Finally, we have studied the decoherence property of the multiple two-level atom system coupled to the thermal radiation fields, which are related to the quantum register.

I. INTRODUCTION

The behavior of the quantum dissipative systems interacting with the background thermal radiation field, is one of the central subjects in quantum statistical physics. Extensive interests in these systems arise from many fields of physics, e.g., condensed matter physics [1], quantum optics [2, 3, 4], quantum measurement [5, 6], quantum computation [7, 8], and so on. The dissipation and decoherence are generated due to the interaction between the system and the thermal bath or reservoir. After the enormous irrelevant degrees of freedom of the thermal bath are integrated out from the von Nuemann equation of the density matrix of the extended system including the environment, the master equation for the reduced density matrix of the relevant system is resulted in some reasonable approximations; for details see, for example, Ref. [2, 3]). For a simple system such as a two-level atom interacting with the background thermal radiation field, as the parameters of the master equation are independent of time, the solutions have been obtained and studied very well [2, 3, 4]. However, for the non-autonomous case where the parameters of the master equation are dependent on time, Ellinas et al. have studied the adiabatic evolution and the corresponding Berry phase in optical resonance [9], and the problem for the general solutions still remains open. For more complicated systems, the solution to the master equation of the reduced density matrix is, in general, difficult to obtain. To solve the problem, the usual way is to convert the master equation to a set of differential equations for some quantum statistical moments or expansion coefficients in terms of some bases truncated at a reasonable order [2, 4]. In the meanwhile, some other useful approximate methods have also been proposed, for instance, the short-time expansion [10], the small lose rate expansion [11], the stochastic unraveling [12], and finally numerical calculations [13], etc. Some exact methods have been also explored [14, 15]. An elegant method was proposed by Briegel and Englert to treat the quantum optical master equations by using the damping bases [16], but the problem was restricted to the autonomous case where the parameters of the master equation are independent of time.

In this paper we shall present a novel algebraic dynamic method to solve the master equations which in many cases are found to have some dynamical algebraic structures. The common feature of the quantum master equations is the existence of the sandwich terms of the Liouville operators where the reduced density matrix of the system is in between some quantum excitation and de-excitation operators, which come from the elimination of the environment degrees of freedom. The sandwich structure of the quantum master equation also appears in thermal field theory where the so-called thermal Lie algebra has been proposed to treat the problem [17, 18], but the parameters of the master equation are still time-independent.

Our new algebraic method is just a generalization of the algebraic dynamical method [19] from quantum mechanical systems to quantum statistical systems with time-dependent parameters. It is designed to treat the sandwich problem and to explore the dynamical algebraic structures of the master equations with time-dependent parameters built in. To this end, the right and left representations [20] as well as the adjoint representations of dynamical algebras are developed and used. This new method has been used successfully to solve the von Nuemann equation for the quantum statistical characteristic function of the two-level Janes-Cummings model [21] and the master equation for the sympathetic cooling of the Bose-Einstein condensate system in the mean field approximation [22]. In this paper, we shall apply this method to solve the master equation of the dissipative two-level atom system in the non-autonomous case, and to study the decoherence of the multiple two level atom system coupled to the radiation thermal bath, which have not been studied before for the non-autonomous case.

The paper is organized as follows. In Sect. II, the model Hamiltonian of the system is given and the master equation for the reduced density matrix of the atom is described. In Sect. III, the dynamical u(2) algebraic structure of the Liouville or rate operator of the master equation is found by introducing the new composite algebras which are constructed from the right and left representations of the relevant algebras, and the dynamical symmetry of the system is established and the integrability of the master equation is thus proved by using the algebraic dynamical theorem [19]. Sect. IV is devoted to the analytical solutions of the master equation for the non-autonomous case where the parameters of the rate operator (or Hamiltonian)are dependent on time, and the approach to the steady solution asymptotically is thus proved. In Sect. V, the dissipation and decoherence of the multiple two-level atom systems are investigated for the non-autonomous case. Discussions and conclusions are given in the final section.

II. DISSIPATIVE TWO-LEVEL ATOM SYSTEM IN NON-AUTONOMOUS CASE

Consider the two-level atom interacting with a thermal quantum radiation field. With the dipole interaction and in the rotating wave approximation, the total system is described by the following model Hamiltonian,

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hat{H}_{bath} + \hbar(\sigma_+\hat{\Lambda} + \sigma_-\hat{\Lambda}^\dagger), \tag{1}$$

where $\hat{H}_{bath} = \sum_k \hbar\omega_k b_k^+ b_k$ describing the background quantum radiation field, $\hat{\Lambda} = \sum_k g_k b_k$ is an operator used to describe the coupling between the atom and the radiation field with coupling constants g_k. Here ω_0 and ω_k are the transition frequency between two levels of the atom and the mode frequencies of the radiation field, respectively. σ_+, σ_-, and σ_z are dimensionless atomic operators obeying the usual Pauli matrix commutation relations [23]. Using the standard technique from quantum optics, one obtains the master equation for the reduced density matrix of the atom [4]

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}(\mathrm{t})}{\mathrm{dt}} &= -\frac{i}{2}\omega_0[\sigma_z, \hat{\rho}(t)] - \frac{\gamma}{2}\ (\bar{n}_0 + 1)\ (\sigma_+\sigma_-\hat{\rho}(t) + \hat{\rho}(t)\sigma_+\sigma_- - 2\sigma_-\hat{\rho}(t)\sigma_+) \\ &\quad -\frac{\gamma}{2}\ \bar{n}_0\ (\sigma_-\sigma_+\hat{\rho}(t) + \hat{\rho}(t)\sigma_-\sigma_+ - 2\sigma_+\hat{\rho}(t)\sigma_-)\end{aligned} \tag{2}$$

where $\bar{n}_0$ is the mean number of photons in the environment and γ denotes the damping rate. They read

$$\begin{aligned}&\bar{n}_0 = [\exp(\hbar\omega_0/k_B T) - 1]^{-1}, \\ &\gamma = 2\pi\sum_k g_k^2\delta(\omega_0 - \omega_k)\end{aligned} \tag{3}$$

Here the term which gives rise to a small Lamb frequency shift $\Delta\omega$ has been neglected. Equation (2) describes an atom interacting with a thermal field at the temperature T. If $T = 0$, then $\bar{n}_0 = 0$.

In the autonomous case, the system has been studied very well. However, it is also interesting to control the system through changing the temperature of the thermal bath, the atomic energy level and the coupling constant. In this case, the parameters of the rate operator $\gamma, \bar{n}_0$, and ω_0 are time-dependent, and the system becomes non-autonomous. More basically, even if the total Hamiltonian of the composite system-the system to be investigated plus the environment, is autonomous, the master equation of the reduced density matrix of the investigated system still becomes non-autonomous under the non-Markovian dynamics [24]. Therefore, the quantum master equation of the reduced density matrix, in general and more rigorously, should be non-autonomous in the sense that its parameters should be time-dependent. To our knowledge, the problem of the non-autonomous systems has not been solved till now. Therefore, in the present paper it is our main goal to solve the problem of the non-autonomous case for the two-level system. In the next section we first study the algebraic structure of the master equation(2) and explore its dynamical symmetry.

III. ALGEBRAIC STRUCTURE OF THE MASTER EQUATION

A. Right and left algebras in the von Neumann space

Following the idea of Ref. [22], the right and left algebras are introduced. First, one notices that the density matrix $\hat{\rho}$ is a super vector in the von Neumann space [20],

$$\hat{\rho} = \sum_{s,s'} \rho_{ss'}|s\rangle\langle s'| \tag{4}$$

where $|s\rangle$ denotes the Fermion state. The σ_+, σ_-, and σ_z can operate on the ket state $|s\rangle$ to the right and on the bra state $\langle s|$ to the left, which form the right and left representations of the usual $su(2)$ algebra as follows [20]

$$su(2)_R : \{\sigma_z^r, \sigma_+^r, \sigma_-^r\}, \quad su(2)_L : \{\sigma_z^l, \sigma_+^l, \sigma_-^l\}. \tag{5}$$

They obey the commutation relations respectively as follows,

$$\begin{aligned} &[\sigma_z^r, \sigma_\pm^r] = \pm 2\sigma_\pm^r, \;\; [\sigma_+^r, \sigma_-^r] = \sigma_z^r, \\ &[\sigma_z^l, \sigma_\pm^l] = \mp 2\sigma_\pm^l, \;\; [\sigma_+^l, \sigma_-^l] = -\sigma_z^l. \end{aligned} \tag{6}$$

It is evident that $su(2)_R$ is isomorphic to the $su(2)$, while $su(2)_L$ is anti-isomorphic to the $su(2)$. This is because that $su(2)_R$ operates, as usual, towards the right on $|s\rangle$. On the other hand, the $su(2)_L$ operates towards the left on $\langle s|$. Since $su(2)_R$ and $su(2)_L$ operate on different spaces(the ket and the bra spaces), they commute each other, i.e.

$$[su(2)_L, su(2)_R] = 0. \tag{7}$$

B. Composite algebra and algebraic structure of the master equation

After having introduced the left and right algebras in Eqs. (5), one can construct the composite $su(2)$ and $u(1)$ algebras as follows,

$$\begin{aligned} su(2) \;&:\; \{\hat{J}_0 = \frac{\sigma_z^r + \sigma_z^l}{2}, \hat{J}_+ = \sigma_+^r \sigma_-^l, \hat{J}_- = \sigma_-^r \sigma_+^l\}. \\ u(1) \;&:\; \hat{U}_0 = \frac{\sigma_z^r - \sigma_z^l}{2} \end{aligned} \tag{8}$$

According to Eqs. (6) it is easy to check the following commutation relations

$$\begin{aligned} &[\hat{J}_0, J_\pm] = \pm 2 J_\pm, \;\; [\hat{J}_+, \hat{J}_-] = \hat{J}_0, \\ &[\hat{U}_0, \hat{J}_\pm] = 0, \quad [U_0, \hat{J}_0] = 0. \end{aligned} \tag{9}$$

The action of the composite $su(2)$ and $u(1)$ algebras on the bases of von-Neumann space is

$$\begin{aligned} \hat{J}_0 |s\rangle\langle s'| \;&=\; \frac{s+s'}{2} |s\rangle\langle s'|, \\ \hat{J}_+ |s\rangle\langle s'| \;&=\; \delta_{s+1,0}\delta_{s'+1,0} |s+2\rangle\langle s'+2|, \\ \hat{J}_- |s\rangle\langle s'| \;&=\; \delta_{s-1,0}\delta_{s'-1,0} |s-2\rangle\langle s'-2|, \\ \hat{U}_0 |s\rangle\langle s'| \;&=\; \frac{s-s'}{2} |s\rangle\langle s'|. \end{aligned} \tag{10}$$

where s and s' are equal to -1 or +1.

Noticing the following identities

$$\sigma_+\sigma_- = \frac{1+\sigma_z}{2}, \quad \sigma_-\sigma_+ = \frac{1-\sigma_z}{2}, \tag{11}$$

and the composite algebra Eqs. (8) introduced above, the master equation(2) can be rewritten as

$$\frac{\mathrm{d}\hat{\rho}(\mathrm{t})}{\mathrm{dt}} = \hat{\Gamma}(t)\hat{\rho}(t), \tag{12}$$

where the rate operator $\hat{\Gamma}$ reads

$$\hat{\Gamma} = -i\omega_0(t)\hat{U}_0 + \gamma(t)\bar{n}_0(t)\hat{J}_+ + \gamma(t)[\bar{n}_0(t)+1]\hat{J}_- - \frac{1}{2}\gamma(t)\hat{J}_0 - \frac{1}{2}\gamma(t)[2\bar{n}_0(t)+1] \tag{13}$$

The master equation (12) is now put into a form similar to the time-dependent Schrödinger equation except that the imaginary number "i" is missing from the left hand side and the rate operator $\hat{\Gamma}$ is non-Hermitian, indicating the dissipative behavior of the system due to the energy exchange with the thermal bath. In Eq. (12), the rate operator $\hat{\Gamma}$ plays the role of the Hamiltonian and the reduced density matrix plays the role of the wave function. Since the rate operator $\hat{\Gamma}$ is a linear function of the $su(2) \oplus u(1)$ generators, the master equation (12) possesses the $su(2) \oplus u(1)$ dynamical symmetry; it is integrable and can be solved analytically according to the algebraic dynamics [19].

IV. EXACT SOLUTION TO THE MASTER EQUATION IN NON-AUTONOMOUS CASE

A. Eigensolutions of the rate operators $\hat{\Gamma}$

In order to better understand the time-dependent solution of the master equation, its decay behavior and its approach to the equilibrium state, we first consider the steady eigenvalue problem of the rate operator $\hat{\Gamma}$ whose eigensolution itself is interesting and peculiar, and contains the steady equilibrium state. The eigen equation reads

$$\hat{\Gamma}\rho(s,s') = \beta(s,s')\rho(s,s'), \tag{14}$$

where $\beta(s,s')$ is the eigenvalue of the rate operator $\hat{\Gamma}$ and (s,s') label the eigenstates in the von Neumann space. This eigenvalue equation is time-independent and can be solved by introducing the following similarity transformation

$$\rho(s,s') = \hat{U}\bar{\rho}(s,s'), \tag{15}$$

where

$$\hat{U} = e^{\alpha_+ \hat{J}_+} e^{\alpha_- \hat{J}_-}, \;\; \hat{U}^{-1} = e^{-\alpha_- \hat{J}_-} e^{-\alpha_+ \hat{J}_+}. \tag{16}$$

Here $\alpha_\pm$ are the parameters specifying the similarity transformation. After some calculations, one has the transformed eigenvalue equation as follows,

$$\begin{aligned} &\bar{\Gamma}\bar{\rho}(s,s') = \beta(s,s')\bar{\rho}(s,s'), \\ &\bar{\Gamma} = \hat{U}^{-1}\hat{\Gamma}\hat{U}, \end{aligned} \tag{17}$$

where the transformed rate operator $\bar{\Gamma}$ is diagonalized and becomes a linear combination of the commuting invariant operators $\hat{J}_0$ and $\hat{U}_0$ which dictate the dynamical symmetry of the system,

$$\bar{\Gamma} = -i\omega_0\hat{U}_0 - \frac{1}{2}\gamma[2(\bar{n}_0+1)\alpha_+ + 1]\hat{J}_0 - \frac{1}{2}\gamma(2\bar{n}_0+1) \tag{18}$$

if the following diagonalization conditions are fulfilled

$$\begin{aligned} &-(\bar{n}_0+1)\alpha_+^2 - \alpha_+ + \bar{n}_0 = 0, \\ &(\bar{n}_0+1)(1+2\alpha_+\alpha_-) + \alpha_- = 0. \end{aligned} \tag{19}$$

The eigenvectors of $\bar{\Gamma}$ are the common solutions of $\hat{J}_0$ and $\hat{U}_0$, just the form of $|s\rangle\langle s'|$ with s and $s' = \pm 1$. Eqs. (19) have two sets of solutions, which yield two sets of eigenvalues $\beta(s,s')$ for the rate operator

$$\begin{aligned} &(a) \qquad \alpha_+ = -1, \;\; \alpha_- = \frac{\bar{n}_0+1}{2\bar{n}_0+1}, \;\; \beta(s,s') = -i\omega_0\frac{s-s'}{2} + \frac{\gamma}{2}(2\bar{n}_0+1)(\frac{s+s'}{2} - 1), \\ &(b) \qquad \alpha_+ = \frac{\bar{n}_0}{\bar{n}_0+1}, \;\; \alpha_- = -\frac{\bar{n}_0+1}{2\bar{n}_0+1}, \;\; \beta(s,s') = -i\omega_0\frac{s-s'}{2} + \frac{\gamma}{2}(2\bar{n}_0+1)(-\frac{s+s'}{2} - 1). \end{aligned} \tag{20}$$

At first glance, it is surprising that two similarity transformations exist to diagonalize the same rate operator and to yield two sets of eigenvalues. This is in contrast to the diagonalization of a Hamiltonian where the unitary transformation to diagonalize the Hamiltonian is usually unique and the set of the eigensolutions is also unique. The peculiar results stem from the special structure of the rate operator (13): (1) it contains a part(from the second to the forth terms) which is a vector in the linear space spanned by the $su(2)$ generators($\hat{J}_+, \hat{J}_-, \hat{J}_0$), and allows two transformations to rotate this part of vector along the $\hat{J}_0$ and $-\hat{J}_0$ directions; (2) the last term of the rate operator is a constant term which is a scalar in the $su(2)$ space and make above two diagonalizing transformation asymmetric. The above two features result in two similarity transformations [22]. However, as will be seen soon, after returning to the physical frame by the inverse transformations, the two sets of eigensolutions coincide. This means that the physical results are objective, independent of the similarity transformations used.

It is interesting to note that both solution (a) and (b) contain the zero-mode steady solution and the nonzero- mode decaying solutions(with negative eigenvalues), which guarantee that any time-dependent solution of the reduced density matrix asymptotically approaches the steady solution as shown below.

Performing an inverse transformation, the eigensolutions of the rate operator is obtained readily

$$\rho(s,s') = \hat{U}\bar{\rho}(s,s') = (1+\alpha_+ J_+)(1+\alpha_- J_-)|s\rangle\langle s'|. \tag{21}$$

Explicitly, both (a) and (b) solutions lead to the same physical eigensolutions.

$$
\begin{aligned}
\beta_1 &= \beta(-1,-1) = 0, & \rho_1 &= \rho(-1,-1) = \frac{\bar{n}_0+1}{2\bar{n}_0+1}|-1\rangle\langle -1| + \frac{\bar{n}_0}{2\bar{n}_0+1}|+1\rangle\langle +1|,\\
\beta_2 &= \beta(+1,+1) = -\gamma(2\bar{n}_0+1), & \rho_2 &= \rho(+1,+1) = |-1\rangle\langle -1| - |+1\rangle\langle +1|,\\
\beta_3 &= \beta(+1,-1) = -\frac{\gamma}{2}(2\bar{n}_0+1) - i\omega_0, & \rho_3 &= \rho(+1,-1) = |+1\rangle\langle -1|,\\
\beta_4 &= \beta(-1,+1) = -\frac{\gamma}{2}(2\bar{n}_0+1) + i\omega_0, & \rho_4 &= \rho(-1,+1) = |-1\rangle\langle +1|. \qquad (22)
\end{aligned}
$$

where the first line of Eqs. (22) is the zero-mode solution corresponding to the steady state.

Another feature of the rate operator is its non-Hermiticity, i.e., $\hat{\Gamma}^\dagger \neq \hat{\Gamma}$, which is evident from $\hat{J}_+^\dagger = \hat{J}_-$, $\hat{J}_-^\dagger = \hat{J}_+$, $\hat{J}_0^\dagger = \hat{J}_0$, and $\hat{U}_0^\dagger = \hat{U}_0$. Since $\hat{\Gamma}$ is non-Hermitian, the eigenvectors of $\hat{\Gamma}$ and $\hat{\Gamma}^\dagger$ constitute a bi-orthogonal basis [25].

Using a similarity transformation $\hat{U}' = e^{-\alpha_+\hat{J}_-}e^{-\alpha_-\hat{J}_+}$ and under the conditions of Eq. (19), diagonalization of the operator $\hat{\Gamma}^\dagger$ can be obtained as follows

$$
\bar{\Gamma}^\dagger = i\omega_0\hat{U}_0 - \frac{1}{2}\gamma[2(\bar{n}_0+1)\alpha_+ + 1]\hat{J}_0 - \frac{1}{2}\gamma(2\bar{n}_0+1) \qquad (23)
$$

In a similar way, we get the eigensolutions of $\hat{\Gamma}^\dagger$.

$$
\begin{aligned}
\beta_1^* &= \beta^*(-1,-1) = 0, & \tilde{\rho}_1 &= \tilde{\rho}(-1,-1) = |-1\rangle\langle -1| + |+1\rangle\langle +1|,\\
\beta_2^* &= \beta^*(+1,+1) = -\gamma(2\bar{n}_0+1), & \tilde{\rho}_2 &= \tilde{\rho}(+1,+1) = \frac{\bar{n}_0}{2\bar{n}_0+1}|-1\rangle\langle -1| - \frac{\bar{n}_0+1}{2\bar{n}_0+1}|+1\rangle\langle +1|,\\
\beta_3^* &= \beta^*(+1,-1) = -\frac{\gamma}{2}(2\bar{n}_0+1) + i\omega_0, & \tilde{\rho}_3 &= \tilde{\rho}(+1,-1) = |+1\rangle\langle -1|,\\
\beta_4^* &= \beta^*(-1,+1) = -\frac{\gamma}{2}(2\bar{n}_0+1) - i\omega_0, & \tilde{\rho}_4 &= \tilde{\rho}(-1,+1) = |-1\rangle\langle +1|. \qquad (24)
\end{aligned}
$$

where β_j^* and $\tilde{\rho}_j$ are the eigenvalues and eigenvectors of the operator $\hat{\Gamma}^\dagger$. It is can be checked that Eqs. (22) and Eqs. (24) are bi-orthogonal.

B. Time-dependent solutions of the master equation in Non-autonomous case

Since Eq. (13) is a linear combination of the generators of the composite algebra, the master equation Eq. (12) possesses the $su(2)\oplus u(1)$ dynamical symmetry and is thus integrable, solvable analytically even in the non-autonomous case [19].

The master equation in the non-autonomous case can be solved by algebraic dynamical method via the following time-dependent gauge transformation which is a generalization of time-independent similarity transformations in the autonomous case to the non-autonomous case (the terminology of gauge transformation is due to the fact that it induces a gauge term in the rate operator similar to the gauge field theory),

$$
\hat{U}_g = e^{\alpha_+(t)\hat{J}_+}e^{\alpha_-(t)\hat{J}_-}. \qquad (25)
$$

After the gauge transformation and under the following gauge conditions

$$
\begin{aligned}
\frac{d\alpha_+(t)}{dt} &= -\gamma(t)[\bar{n}_0(t)+1]\alpha_+^2(t) - \gamma(t)\alpha_+(t) + \gamma(t)\bar{n}_0(t) = -\gamma(t)[\bar{n}_0(t)+1][\alpha_+(t)+1][\alpha_+(t) - \frac{\bar{n}_0(t)}{\bar{n}_0(t)+1}]\\
\frac{d\alpha_-(t)}{dt} &= \gamma(t)[\bar{n}_0(t)+1][1+2\alpha_+(t)\alpha_-(t)] + \gamma(t)\alpha_-(t), \qquad (26)
\end{aligned}
$$

the gauged rate operator is diagonalized and the gauged master equation becomes simple and integrable,

$$
\begin{aligned}
\frac{d\bar{\rho}(t)}{dt} &= \bar{\Gamma}(t)\bar{\rho}(t),\\
\bar{\Gamma}(t) &= \hat{U}_g^{-1}\Gamma(t)\hat{U}_g - \hat{U}_g^{-1}\frac{d\hat{U}_g}{dt} = -i\omega_0(t)\hat{U}_0 - \frac{1}{2}\gamma(t)\{2[\bar{n}_0(t)+1]\alpha_+(t)+1]\hat{J}_0 - \frac{1}{2}\gamma(t)[2\bar{n}_0(t)+1]\}. \qquad (27)
\end{aligned}
$$

which is a linear function of the complete set of the commuting operators (invariant operators)$\hat{U}_0$ and $\hat{J}_0$, and clearly shows a $u(2)$ dynamical symmetry. The solution of Eqs. (27) reads

$$\bar{\rho}(t) = e^{\int_0^t \bar{\Gamma}(\tau)d\tau}\bar{\rho}(0), \tag{28}$$

For the initial conditions

$$\begin{aligned} \alpha_+(0) &= \alpha_-(0) = 0, \\ or\ \rho(0) &= \bar{\rho}(0) = \sum_{ss'} p_{ss'}|s\rangle\langle s'| \end{aligned} \tag{29}$$

or , we finally obtain the solution,

$$\rho(t) = e^{\alpha_+(t)\hat{J}_+}e^{\alpha_-(t)\hat{J}_-}e^{-i\hat{U}_0\int_0^t \omega_0(\tau)d\tau - \hat{J}_0\int_0^t \gamma(\tau)((\bar{n}_0(\tau)+1)\alpha_+(\tau)+\frac{1}{2})d\tau - \int_0^t \frac{\gamma(\tau)}{2}(2\bar{n}_0(\tau)+1)d\tau}\rho(0). \tag{30}$$

Once the reduced density matrix of the non-autonomous system is obtained, the averages of the physical observables σ_z, σ_+, and σ_+ can be calculated. For the system initially in a pure state, we have $\rho(0) = |\mu|^2|1\rangle\langle 1| + |\nu|^2|-1\rangle\langle -1| + \mu\nu^*|1\rangle\langle -1| + \mu^*\nu|-1\rangle\langle 1|$, where $|\mu|^2 + |\nu|^2 = 1$. Then one obtains

$$\begin{aligned} \rho(t) &= [f_{1,1}(t)|\mu|^2(1+\alpha_+(t)\alpha_-(t)) + f_{-1,-1}(t)|\nu|^2\alpha_+(t)]|1\rangle\langle 1| + [f_{1,1}(t)|\mu|^2\alpha_-(t) + f_{-1,-1}(t)|\nu|^2]|-1\rangle\langle -1| \\ &\quad + f_{1,-1}(t)\mu\nu^*|1\rangle\langle -1| + f_{-1,1}(t)\mu^*\nu|-1\rangle\langle 1|, \end{aligned} \tag{31}$$

where

$$f_{s,s'}(t) = e^{-i\frac{s-s'}{2}\int_0^t \omega_0(\tau)d\tau - \frac{s+s'}{2}\int_0^t \gamma(\tau)\{[\bar{n}_0(\tau)+1]\alpha_+(\tau)+\frac{1}{2}\}d\tau - \frac{1}{2}\int_0^t \gamma(\tau)[2\bar{n}_0(\tau)+1]d\tau}, \tag{32}$$

From Eq. (31), we get

$$\begin{aligned} \langle\sigma_z\rangle &= f_{1,1}(t)|\mu|^2[1+\alpha_+(t)\alpha_-(t) - \alpha_-(t)] + f_{-1,-1}(t)|\nu|^2[\alpha_+(t) - 1], \\ \langle\sigma_+\rangle &= f_{-1,1}(t)\mu^*\nu \\ \langle\sigma_-\rangle &= f_{1,-1}(t)\mu\nu^*. \end{aligned} \tag{33}$$

For the autonomous case, $\bar{n}_0$ and γ are independent of time, α_+ and α_- have the following analytical solutions,

$$\begin{aligned} \alpha_+(t) &= \frac{1 - e^{-\gamma(2\bar{n}_0+1)t}}{\frac{\bar{n}_0+1}{\bar{n}_0} + e^{-\gamma(2\bar{n}_0+1)t}}, \\ \alpha_-(t) &= \frac{(\bar{n}_0+1)\bar{n}_0[\frac{\bar{n}_0+1}{\bar{n}_0} + e^{-\gamma(2\bar{n}_0+1)t}][1 - e^{-\gamma(2\bar{n}_0+1)t}]}{(2\bar{n}_0+1)^2 e^{-\gamma(2\bar{n}_0+1)t}}. \end{aligned} \tag{34}$$

Then

$$\begin{aligned} f_{1,1}(t) &= \frac{(2\bar{n}_0+1)e^{-\gamma(2\bar{n}_0+1)t}}{(\bar{n}_0+1) + \bar{n}_0 e^{-\gamma(2\bar{n}_0+1)t}}, \\ f_{-1,-1}(t) &= \frac{(\bar{n}_0+1) + \bar{n}_0 e^{-\gamma(2\bar{n}_0+1)t}}{2\bar{n}_0+1}, \\ f_{1,-1}(t) &= e^{-i\omega_0 t - \frac{\gamma}{2}(2\bar{n}_0+1)t}, \\ f_{-1,1}(t) &= e^{i\omega_0 t - \frac{\gamma}{2}(2\bar{n}_0+1)t}. \end{aligned} \tag{35}$$

Inserting Eqs. (34) and Eqs. (35) into Eqs. (33), we recover the well-known results for the autonomous system.

C. Approaching the steady solution

Like the autonomous case, the time-dependent solutions of the master equation in the non-autonomous case also asymptotically approaches the steady solution satisfying

$$\frac{\mathrm{d}\hat{\rho}(\mathrm{t})}{\mathrm{dt}} = \hat{\Gamma}\hat{\rho}(t) = 0 \tag{36}$$

which has the same solution as the zero-mode eigensolutions of $\hat{\Gamma}$, namely the steady solutions with the parameters α_+ and α_- obeying

$$\frac{d\alpha_+(t)}{dt} = 0,$$
$$\frac{d\alpha_-(t)}{dt} = 0. \tag{37}$$

whose solutions are obviously the same as Eqs. (19) and should have the two same sets of solutions (a) and (b) as given in Eqs. (20). For the autonomous case, the two similarity transformations as tools to diagonalize the rate operator are on the equal footing and generate the same physical solution as proved above; while for the non-autonomous case, the properties of the two sets of solutions of α_+ and α_- should be examined on the background of their time evolution. It is found from the Eq. (26) that if $\alpha_+ = -1-\epsilon$,then $\frac{d\alpha_+(t)}{dt} < 0$ and α_+ will go away further from -1 towards negative direction; while as $\alpha_+ = -1+\epsilon$, then $\frac{d\alpha_+(t)}{dt} > 0$ and α_+ will go away further from -1 towards positive direction. Thus the steady solution $\alpha_+ = -1$ is unstable. Since the solution of $\alpha_-(t)$ depends on $\alpha_+(t)$, it is also unstable. Therefore the solution (a) is unstable and can not be reached from the initial condition $\alpha_+ = \alpha_- = 0$. Instead, the solution (b) is partly stable in the following sense. Because the first equation of Eq. (26) reads

$$\frac{d\alpha_+(t)}{dt} = -\gamma(t)[\bar{n}_0(t)+1][\alpha_+(t)+1][\alpha_+(t) - \frac{\bar{n}_0(t)}{\bar{n}_0(t)+1}]. \tag{38}$$

It's easy to see that

$$\begin{cases} d\alpha_+(t)/dt > 0, \text{ if } 0 < d\alpha_+(t) < \frac{\bar{n}_0(t)}{\bar{n}_0(t)+1} \\ d\alpha_+(t)/dt < 0, \text{ if } \alpha_+(t) > \frac{\bar{n}_0(t)}{\bar{n}_0(t)+1} \text{ and } \alpha_+(t) < -1 \end{cases}$$

$d\alpha_+(t)/dt > 0(<0)$ if $d\alpha_+(t)/dt > 0$ (if $\alpha_+(t) > \frac{\bar{n}_0(t)}{\bar{n}_0(t)+1}$ and $\alpha_+(t) < -1$). With the initial condition $\alpha_+ = 0$, we see that α_+ approaches the value $\frac{\bar{n}_0(\infty)}{\bar{n}_0(\infty)+1} = \frac{\bar{n}_0}{\bar{n}_0+1}$ asymptotically from zero. However, $\alpha_-(t)$ can not reach its steady value $-\frac{\bar{n}_0+1}{2\bar{n}_0+1}$. To study the asymptotic behavior of α_-, we define $y(t) = \alpha_-(t)\times\exp\{-\int_0^t \gamma(\tau)(\bar{n}_0(\tau)+1)(\alpha_+(\tau)+1)d\tau\} = \alpha_-(t)\exp\int_0^t p(\tau)d\tau$. The time differential of $y(t)$ is given by $b\exp\int_0^t p(\tau)d\tau$ where $b = \dot{\alpha}_-(t) + \alpha_-(t)p(t)$. Since $b \longrightarrow \gamma(\bar{n}_0+1)$ which is bounded and $p(t)$ is negative for large t, the differential tends to zero and, hence, $y(t)$ is towards a constant. This implies that $\alpha_-(t)$ diverges asymptotically. So the unique solution of Eq. (26) has the following asymptotic properties:

$$\alpha_+(\infty) = \frac{\bar{n}_0}{\bar{n}_0+1},$$
$$\alpha_-(t)\times e^{-\int_0^t \gamma(\tau)(\bar{n}_0(\tau)+1)(\alpha_+(\tau)+1)d\tau}\,|_{t\to\infty} = const. \tag{39}$$

Using the above asymptotic relations, one obtains the asymptotic results of the time-dependent solutions as follows

$$\begin{aligned}
\rho_{+-}(t)\,|_{t\to\infty} &= e^{-i\int_0^t \omega_0(\tau)d\tau - \int_0^t (\gamma(\tau)(\bar{n}_0(\tau)+\frac{1}{2})d\tau} e^{\alpha_+(t)\hat{J}_+} e^{\alpha_-(t)\hat{J}_-} |+1\rangle\langle -1| \\
&= e^{-i\int_0^t \omega_0(\tau)d\tau - \int_0^t \gamma(\tau)(\bar{n}_0(\tau)+\frac{1}{2})d\tau} |+1\rangle\langle -1| \longrightarrow 0, \\
\rho_{-+}(t)\,|_{t\to\infty} &= e^{+i\int_0^t \omega_0(\tau)d\tau - \int_0^t \gamma(\tau)(\bar{n}_0(\tau)+\frac{1}{2})d\tau} e^{\alpha_+(t)\hat{J}_+} e^{\alpha_-(t)\hat{J}_-} |-1\rangle\langle +1| \\
&= e^{+i\int_0^t \omega_0(\tau)d\tau - \int_0^t \gamma(\tau)(\bar{n}_0(\tau)+\frac{1}{2})d\tau} |+1\rangle\langle -1| \longrightarrow 0, \\
\rho_{++}(t)\,|_{t\to\infty} &= e^{-\int_0^t \gamma(\tau)(\bar{n}_0(\tau)+1)(\alpha_+(\tau)+1)d\tau} e^{\alpha_+(t)\hat{J}_+} e^{\alpha_-(t)\hat{J}_-} |+1\rangle\langle +1| \\
&= e^{-\int_0^t \gamma(\tau)(\bar{n}_0(\tau)+1)(\alpha_+(\tau)+1)d\tau} [|+1\rangle\langle +1| + \alpha_-(t)(|-1\rangle\langle -1| + \alpha_+(t)|+1\rangle\langle +1|)] \\
&\longrightarrow const. \times (|-1\rangle\langle -1| + \frac{\bar{n}_0}{\bar{n}_0+1}|+1\rangle\langle +1|) = const. \times \rho_{steady}, \\
\rho_{--}(t)\,|_{t\to\infty} &= e^{-\int_0^t \gamma(\tau)\{-(\bar{n}_0(\tau)+1))\alpha_+(\tau)+\bar{n}_0(\tau)\}d\tau} e^{\alpha_+(t)\hat{J}_+} e^{\alpha_-(t)\hat{J}_-} |-1\rangle\langle -1| \\
&= e^{-\int_0^t \gamma(\tau)[-(\bar{n}_0(\tau)+1)\alpha_+(\tau)+\bar{n}_0(\tau)]d\tau} [|-1\rangle\langle -1| + \alpha_+(t)|+1\rangle\langle +1|] \longrightarrow c\times\rho_{steady}.
\end{aligned} \tag{40}$$

In the above derivation, we have used the following asymptotic relations:

$$e^{-\int_0^t \gamma(\tau)[-(\bar{n}_0(\tau)+1)(\alpha_+(\tau)+\bar{n}_0(\tau)]d\tau}\,|_{t\to\infty}\longrightarrow 1. \tag{41}$$

which can be proved by Eq. (26).

The above results indicate that the time-dependent solutions of the master equation in the non-autonomous case also asymptotically approach the steady solution irrespective of their initial conditions, and that the divergent behavior of $\alpha_-(t)$ and eq.(37) play an important role in the process of approaching equilibrium state.

V. DECOHERENCE OF MULTIPLE ATOM SYSTEMS

It is straightforward to generalize the above model to N two-level atoms. The problem of the N two-level atoms system is related to the quantum register and the entanglement state in quantum computation [26]. Because of the inevitable coupling of the atoms to the external environment, the entanglement state will lose the coherence among different atomic states and some information carried by the multiple atoms will be lost. Palma et al. has studied the impact of decoherence on the efficiency of the Shor quantum algorithm and the decoherence of quantum register at the two qubit level [27]. However, the solution to the problem in the non-autonomous case is still lacking. The N two-level atoms coupled to the quantum radiation environment can be described by the following model Hamiltonian,

$$\begin{aligned}
\hat{H} &= \hat{H}_s + \hat{H}_{env} + \hat{H}_I, \\
\hat{H}_s &= \frac{1}{2}\sum_{k=1}^{N}\hbar\Omega_k\sigma_k^z, \\
\hat{H}_I &= \sum_{k=1}^{N}\sum_{j=1}^{\infty}(g_{kj}b_j^\dagger\sigma_k^- + h.c.).
\end{aligned} \tag{42}$$

Taking the same procedure and the same approximations as those for the one atom case , the master equation for the N two-level atoms can be obtained

$$\begin{aligned}
\dot{\rho}_N &= \sum_{k=1}^{N}\Gamma^k\rho_N = \Gamma\rho_N \\
\Gamma &= \sum_{k=1}^{N}\Gamma^k, \\
\Gamma^k &= -i\omega_0\hat{U}_0^k + \gamma\bar{n}_0\hat{J}_+^k + \gamma(\bar{n}_0+1)\hat{J}_-^k - \frac{\gamma}{2}\hat{J}_0^k - \frac{\gamma}{2}(2\bar{n}_0+1).
\end{aligned} \tag{43}$$

Here we have assumed that the N two level atoms are identical and coupled to the same environment so that the decay rate γ for different atoms and the mean number of environment photons $\bar{n}_0$ are the same (if they are different, a superscript "k" should be put on each pair of parameters, namely γ^k and $\bar{n}_0^k$). We have also set $\Omega_1 = \Omega_2 = \cdots = \Omega_N = \omega_0$. The solution to Eqs. (43) reads

$$\rho_N(t) = \prod_{k=1}^{N}\rho_k(t). \tag{44}$$

According to Eq. (30), the density matrix $\rho_k(t)$ of the k-th qubit is

$$\begin{aligned}
\rho_k(t) &= e^{\alpha_+^k(t)\hat{J}_+^k}e^{\alpha_-^k(t)\hat{J}_-^k}e^{\int_0^t\bar{\Gamma}^k(\tau)d\tau}\rho_k(0), \\
\bar{\Gamma}^k(t) &= -i\omega_0(t)\hat{U}_0^k - \frac{1}{2}\gamma(t)\{2[\bar{n}_0(t)+1]\alpha_+^k(t)+1\}\hat{J}_0^k \\
&\quad -\frac{1}{2}\gamma(t)[2\bar{n}_0(t)+1].
\end{aligned} \tag{45}$$

where $\bar{\Gamma}^k(t) = -i\omega_0(t)\hat{U}_0^k - \frac{1}{2}\gamma(t)\{2[\bar{n}_0(t)+1]\alpha_+^k(t)+1\}\hat{J}_0^k - \frac{1}{2}\gamma(t)[2\bar{n}_0(t)+1]$. As in the last section, the initial density matrix $\rho_k(0)$ of each qubit can be expanded in terms of the superbases, $\rho_k(0) = \sum_{(s,s')}c_{s,s'}^k|s\rangle^{kk}\langle s'|$. Here

$\alpha_+^k(t)$ and $\alpha_-^k(t)$ obey the Eqs. (26), because we have assumed that the parameters of the rate operators for different atoms are the same functions of time. If γ^k and $\bar{n}_0^k$ are different for different atoms, $\alpha_+^k(t)$ and $\alpha_-^k(t)$ are also different for different atoms. However they obey the same form of Eqs. (26) but with different parameters γ^k and $\bar{n}_0^k$. To illuminate the model concretely, we also consider the two qubit system with the initial state in the pure state, namely $|\psi(0)\rangle = \alpha|+-\rangle + \beta|-+\rangle$, and

$$\begin{aligned}\rho_2(0) \;=\;& |\alpha|^2|+1\rangle^{11}\langle+1|\otimes|-1\rangle^{22}\langle-1|+|\beta|^2|-1\rangle^{11}\langle-1|\otimes|+1\rangle^{22}\langle+1|+\alpha\beta^*|+1\rangle^{11}\langle-1|\otimes|-1\rangle^{22}\langle+1|\\ &+\alpha^*\beta|-1\rangle^{11}\langle+1|\otimes|+1\rangle^{22}\langle-1|\end{aligned} \tag{46}$$

The time-dependent solution of the density matrix is now

$$\begin{aligned}\rho_2(t) \;=\;& |\alpha|^2 f_{1,1}(t)[(1+\alpha_+(t)\alpha_-(t))|+1\rangle^{11}\langle+1|+\alpha_-(t)|-1\rangle^{11}\langle-1|]\otimes f_{-1,-1}(t)\,[|-1\rangle^{22}\langle-1|+\alpha_+(t)|+1\rangle^{22}\langle+1|]\\ &+|\beta|^2 f_{-1,-1}(t)\,[|-1\rangle^{11}\langle-1|+\alpha_+(t)|+1\rangle^{11}\langle+1|]\otimes f_{1,1}(t)[(1+\alpha_+(t)\alpha_-(t))|+1\rangle^{22}\langle+1|+\alpha_-(t)|-1\rangle^{22}\langle-1|]\\ &+\alpha\beta^* f_{1,-1}|+1\rangle^{11}\langle-1|\otimes f_{-1,1}|-1\rangle^{22}\langle+1|+\alpha^*\beta f_{-1,1}|-1\rangle^{11}\langle+1|\otimes f_{1,-1}|+1\rangle^{22}\langle-1|\end{aligned} \tag{47}$$

where $f_{s,s'}(t)$ are the same as given in Eqs. (32). For the autonomous case, $\gamma(t)$, $\bar{n}_0(t)$ and $\omega_0(t)$ are independent of time, and the solutions of $\alpha_+(t)$ and $\alpha_-(t)$ are determined from Eqs. (26), which are the same as Eqs. (34). Eq. (47) is now reduced to the result for the autonomous case, it reads

$$\begin{aligned}\rho_2(t) \;=\;& e^{-\gamma(2\bar{n}_0+1)t}\{2\frac{\bar{n}_0|\alpha|^2-(\bar{n}_0+1)|\beta|^2}{2\bar{n}_0+1}\rho_1^1\otimes\rho_2^2+2\frac{\bar{n}_0|\beta|^2-(\bar{n}_0+1)|\alpha|^2}{2\bar{n}_0+1}\rho_2^1\otimes\rho_1^2+\alpha\beta^*\rho_3^1\otimes\rho_4^2+\alpha^*\beta\rho_4^1\otimes\rho_3^2\}\\ &-e^{-2\gamma(2\bar{n}_0+1)t}4\frac{\bar{n}_0(\bar{n}_0+1)}{(2\bar{n}_0+1)^2}\rho_2^1\otimes\rho_2^2+\rho_1^1\otimes\rho_1^2.\end{aligned} \tag{48}$$

The above solution indicates that during the time evolution, the density matrix of the two qubit entanglement state will approach the steady density matrix(the last term in Eq. (48))and lose its coherence. The characteristic time of the decoherence is $\tau_{decoh} = \frac{1}{\gamma(2\bar{n}_0+1)}$.

VI. SUMMARY

In this paper, Based on the quantum master equation, we have investigated the dissipative and decoherence behaviors of the two-level atom system coupled to the environment of thermal quantum radiation fields. The dynamical u(2) algebraic structure of the quantum master equation of the two-level dissipative system in the non-autonomous case is found by virtue of left and right algebras. By the algebraic dynamical method and proper gauge transformations, the analytical solutions to the non-autonomous master equation are obtained and the long time behavior of the system has been examined. Finally we extended the model to the multiple two-level dissipative atom system and its decoherence is studied in terms of the density matrices for the non-autonomous case, which are given analytically and related to quantum register and quantum computation. Since the master equations of a wide class of dissipative quantum systems possess some dynamical algebraic structures, the present method used by us may serve as a useful tool in quantum statistical physics to treat the dissipative and decoherence problems. In addition, the results obtained in this paper may be practically useful for the analysis of the decoherence of the multiple two-level atom systems and quantum register.

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